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On extensions of a theorem of Baxter

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Dedicated to Barry Simon on the occasion of his 60th birthday

Abstract

We combine the Riemann–Hilbert approach with the techniques of Banach algebras to obtain an extension of Baxter's Theorem for polynomials orthogonal on the unit circle. This is accomplished by using the link between the negative Fourier coefficients of the scattering function and the coefficients in the recurrence formula satisfied by these polynomials.

1. Introduction

Recently, there has been a upsurge in interest in the theory of orthogonal polynomials on the unit circle (OPUC). To a large extent this can be attributed to the two volumes by Barry Simon devoted to the theory of these polynomials. If μ is a positive probability measure supported on the unit circle with an infinite number of points of increase then a sequence of polynomials $\varphi_n(z) = \kappa_n z^n + \cdots$ of degree n, $n = 0, 1, \ldots$ satisfying

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(e^{i\theta}) \overline{\varphi_m(e^{i\theta})}\, d\mu = \delta_{n,m}$$

defines a set of orthonormal polynomials on the unit circle $\mathbb{T}$. We will assume that the leading coefficient κ_n in each polynomial φ_n is positive, so the above equation uniquely specifies the polynomials. It is well known [6,14,15] that these polynomials satisfy the following recurrence

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formula:

$$\boldsymbol{\varphi}_{n+1}(z) = A_n(z)\boldsymbol{\varphi}_n(z), \tag{1}$$

with

$$\boldsymbol{\varphi}_n(z) = \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix}, \quad \boldsymbol{\varphi}_0(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$A_n(z) = \rho_n^{-1} \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix}. \tag{2}$$

In the formula above, $\varphi_n^*(z) = z^n \bar{\varphi}_n(1/z)$ is called the *reversed* polynomial. The α_n are known as the *recurrence* or *Verblunsky coefficients* [14], and have the property that $|\alpha_n| < 1$. Also $\rho_n = (1 - |\alpha_n|^2)^{1/2} = k_n/k_{n+1}$. It is well known that for any $n \in \mathbb{N}$, φ_n has all its zeros strictly inside the unit circle.

Let us recall (see [14, Chapter 5]) that a *Beurling weight* is a two-sided sequence $v = \{v(n)\}_{-\infty}^{\infty}$ with the properties

$$v(0) = 1, \quad v(n) \geqslant 1, \tag{3}$$

$$v(n) = v(-n), \tag{4}$$

$$v(n+m) \leqslant v(n)v(m). \tag{5}$$

These properties imply the existence of the limit

$$\lim_{n \to +\infty} v(n)^{1/n} = R \geqslant 1. \tag{6}$$

If $R = 1$, then v is a *strong* Beurling weight.

Each Beurling weight v has associated the Banach spaces ℓ_v of two-sided sequences $f = \{f_n\}_{n=-\infty}^{+\infty}$ with

$$\|f\|_v \stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} v(n)|f_n| < \infty;$$

this norm extends naturally to any one-sided sequence by completing the latter with zeros.

In what follows for a positive measure μ on $\mathbb{T}$ we will write

$$d\mu = w(\theta)\, d\theta + d\mu_s, \tag{7}$$

after the Lebesgue–Radon–Nikodym decomposition theorem. We will abuse notation using $w(\theta)$ and $w(e^{i\theta})$ interchangeably, expecting that this will not cause confusion.

As noted by Simon [14], Baxter's Theorem which relates the decay rate of the recurrence coefficients $\alpha = \{\alpha_n\}_{n=0}^{+\infty}$ to the decay rate of the Fourier coefficients $c = \{c_n\}_{n=-\infty}^{+\infty}$ of the log w,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \log w(\theta)\, d\theta, \quad n \in \mathbb{Z}, \tag{8}$$

plays an important role in the theory.

Theorem 1 (*Baxter [1]*). *Suppose v is a strong Beurling weight ($R = 1$). Then, $\alpha \in \ell_v$ if and only if $\mu_s = 0$, $w \in C([-\pi, \pi])$ is positive, and $c \in \ell_v$.*

In the summer of 2003 Barry Simon asked one of us (JSG) whether it was possible to extend the above Theorem for the case of Beurling weights with $R > 1$ as had been done for polynomials orthogonal on the real line (see Geronimo [4]). What is needed is to remove poles from the weight function and to be able to control the decay of the coefficients of the new weight function. For polynomials orthogonal on the real line this is accomplished by the Christoffel–Uvarov formulas (see [15,16,4]). However, such formulas are not available at present in the case of the unit circle. The necessary control over the coefficients, however, can be established using Banach algebra techniques coupled with those coming from the Riemann–Hilbert approach (see [2,8], or for an alternate approach [13]). We begin by collecting the results from scattering theory that will be needed. In particular, we introduce the scattering function $\mathcal{S}$ and show that its projection $\mathcal{P}_-(\mathcal{S})$ (see below) governs the rate of decay α. We then show that the product of two weights whose coefficients decay exponentially with the same rate give a weight whose coefficients decay with the same rate. Next we apply this to Berstein–Szegő perturbations of weights. Finally, we obtain a generalization of Baxter's Theorem.

2. A generalization of Baxter's theorem

As was noticed by Baxter [1], Banach algebras may be associated with the Beurling weights v in the following manner. Let $a,\ b \in \ell_v$; then their convolution is given by

$$(a * b)(n) = \sum_{k=-\infty}^{\infty} a(k)b(n-k),$$

which is absolutely convergent by (3) and (5),

$$\|a * b\|_v = \sum_n v(n) \sum_k |a(k)b(n-k)| \leqslant \|a\|_v \|b\|_v. \tag{9}$$

Thus, if we consider the space of functions

$$\mathcal{A}_v \stackrel{\text{def}}{=} \left\{ f(z) = \sum_{k\in\mathbb{Z}} f_k z^k : \|f\|_v = \sum_k v(k)|f_k| < \infty \right\}, \tag{10}$$

Eq. (9) shows that it is closed under multiplication and so forms an algebra. For $v \equiv 1$ we obtain the Wiener algebra $\mathcal{A}_1$, containing $\mathcal{A}_v$ for any Beurling weight v.

On the set of all two-sided sequences $\{d(n)\}_{n\in\mathbb{Z}}$ we define the projectors $\mathcal{P}_-$ and $\mathcal{P}_+$:

$$(\mathcal{P}_- d)(n) = \begin{cases} d(n) & \text{if } n \leqslant 0, \\ 0 & \text{if } n > 0 \end{cases} \quad \text{and} \quad (\mathcal{P}_+ d)(n) = \begin{cases} d(n) & \text{if } n \geqslant 0, \\ 0 & \text{if } n < 0. \end{cases}$$

These projectors can be naturally extended to A_v, and give rise to two subalgebras associated with $\mathcal{A}_v$: $\mathcal{A}_v^+ \stackrel{\text{def}}{=} \mathcal{P}_+(\mathcal{A}_v)$ and $\mathcal{A}_v^- \stackrel{\text{def}}{=} \mathcal{P}_-(\mathcal{A}_v)$. In other words, $\mathcal{A}_v^\pm$ are the set of functions $f \in \mathcal{A}_v$ whose Fourier coefficients f_n vanish for $n < 0$ or $n > 0$, respectively. It is easy to see because of (3) that if $f \in \mathcal{A}_v$ then $f(z)$ is continuous for $1/R \leqslant |z| \leqslant R$ (which is the maximal ideal space associated to $\mathcal{A}_v$) and analytic for $1/R < |z| < R$. Likewise, if f is in $\mathcal{A}_v^+$ then f is continuous for $|z| \leqslant R$ and analytic for $|z| < R$ (in which case the series in (10) is its Maclaurin expansion convergent at least in this disk), and if f is in $\mathcal{A}_v^-$ then f is continuous for $|z| \geqslant 1/R$ and analytic for $|z| > 1/R$.

If we denote by $\hat{\varphi}_n(z) \stackrel{\text{def}}{=} z^{-n}\varphi_n(z)$, then $\hat{\varphi}_n \in \mathcal{A}_v^-$, and $\varphi_n^* \in \mathcal{A}_v^+$, and we may recast (1) as

$$G_{n+1}(z) = B_n(z)G_n, \tag{11}$$

where

$$G_n(z) = \begin{pmatrix} \hat{\varphi}_n(z) \\ \varphi_n^*(z) \end{pmatrix} \quad \text{and} \quad B_n(z) = \rho_n^{-1} \begin{pmatrix} 1 & -z^{-(n+1)}\bar{\alpha}_n \\ -\alpha_n z^{n+1} & 1 \end{pmatrix}.$$

The next lemma is implicitly given by Baxter [1] and Simon [14, Chapter 5]. We present its proof for completeness of the reading.

Lemma 1. *Suppose v is a Beurling weight, and $\{\alpha\}_0^\infty \in \ell_v$. Then*

$$\lim_n \varphi_n^* = f_+ \in \mathcal{A}_v^+, \tag{12}$$

$$\lim_n \hat{\varphi}_n = f_- \in \mathcal{A}_v^-, \tag{13}$$

where convergence is understood in the $\|\cdot\|_v$ norm. For $|z| = 1$, $f_-(z) = \overline{f_+(z)} = \bar{f}_+(1/z)$.

Furthermore, in the decomposition (7), $d\mu_s = 0$, and

$$w = \frac{1}{f_+ f_-} \tag{14}$$

on $\mathbb{T}$; hence, w is a positive continuous function on $\mathbb{T}$.

Proof. Let us denote by $\Phi_n(z) = \kappa_n^{-1}\varphi_n(z)$ the monic orthogonal polynomial, and $\hat{\Phi}_n(z) = z^{-n}\Phi_n(z)$. Then from Eq. (11) we find

$$\begin{aligned} \Phi_{n+1}^*(z) &= \Phi_n^*(z) - \alpha_n z^{n+1}\hat{\Phi}_n(z), \\ \hat{\Phi}_{n+1}(z) &= \hat{\Phi}_n(z) - \overline{\alpha_n} z^{-(n+1)}\Phi_n^*(z). \end{aligned} \tag{15}$$

A consequence of (5) is that for $f \in \mathcal{A}_v$ and $k \in \mathbb{Z}$, $\|z^k f(z)\|_v \leqslant v(k)\, \|f\|_v$. Thus,

$$\|\Phi_{n+1}^*\|_v \leqslant \|\Phi_n^*\|_v + v(n+1)|\alpha_n|\, \|\hat{\Phi}_n\|_v.$$

A similar formula holds for $\hat{\Phi}_{n+1}$, so that by induction it follows that

$$\|g_{n+1}\|_v \leqslant \prod_{i=0}^{n} (1 + v(i+1)|\alpha_i|),$$

with $g_n = \Phi_n^*$ or $\hat{\Phi}_n$, so that $\sup_n \|\Phi_n^*\|_v < \infty$, $\sup_n \|\hat{\Phi}_n\|_v < \infty$.

By (15),

$$\|\Phi_n^* - \Phi_m^*\|_v \leqslant \sum_{i=m}^{n-1} v(i+1)|\alpha_i|\|\hat{\Phi}_i\|_v,$$

which yields that $\{\Phi_n^*\}$ is a Cauchy sequence in $\mathcal{A}_v^+$, and hence has a limit in $\mathcal{A}_v^+$. Since

$$\kappa_n = \prod_{i=0}^{n-1} \left(1 - |\alpha_i|^2\right)^{-1/2}$$

converge, (12) follows. Similar reasoning gives (13).

Since for $|z| = 1$, $\hat{\varphi}_n = \overline{\varphi_n^*}$ we find for $|z| = 1$ that $f_-(z) = \overline{f_+(z)} = \bar{f}_+(1/z)$.
On the other hand, by (12)–(13),

$$\lim_n \||\varphi_n^*|^2 - f_- f_+\|_v = 0.$$

Using the well known fact from the theory of OPUC [6,14,15] that $\varphi_n^*(z) \neq 0$, $|z| \leqslant 1$, and for any $h \in C(\mathbb{T})$, where $C(\mathbb{T})$ is the set of continuous functions on the unit circle,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,\frac{d\theta}{|\varphi_n^*(e^{i\theta})|^2} \to \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)d\mu(\theta)$$

the result follows. □

Remark 1. We may recast the last result in terms more familiar to the orthogonal polynomials community. The *Szegő function* of w (see e.g. [15, Chapter X, Section 10.2]) is defined by

$$D(z) \stackrel{\text{def}}{=} \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log\, w(e^{i\theta})\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right).$$

This function is piecewise analytic and non-vanishing for $|z| \neq 1$, and we denote by D_i and D_e its values for $|z| < 1$ and $|z| > 1$, respectively. Then

$$f_+(z) = D_\mathrm{i}^{-1}(z), \quad f_-(z) = D_\mathrm{e}(z).$$

Also the last identity of Lemma 1 is consistent with the property

$$\overline{D_\mathrm{i}\left(\frac{1}{\bar{z}}\right)} = \frac{1}{D_\mathrm{e}(z)}, \quad |z| > 1.$$

Lemma 1 and Eq. (14) show that $1/w \in \mathcal{A}_v$, which gives a natural meromorphic extension of w such that $w \neq 0$ for $R^{-1} \leqslant |z| \leqslant R$, and this is the meaning we will be giving to w in what follows.

From the previous works [2,5,8,14] it is clear that the function,

$$\mathcal{S} = \frac{f_-}{f_+} = D_\mathrm{i} D_\mathrm{e}, \tag{16}$$

plays a prominent role in the theory. By Lemma 1, we have that $|\mathcal{S}(z)| = 1$ for $|z| = 1$. Following Faddeyev's classic article on inverse scattering [3] and Newton's book on Scattering theory [10] we call the unimodular function $\mathcal{S}$ the *scattering function*.

Straightforward computation shows that

$$\mathcal{S}(z) = \exp\left(\sum_{k=1}^{+\infty}\left(c_k z^k - c_{-k} z^{-k}\right)\right) = \exp\left(\sum_{k=1}^{+\infty}\left(c_k z^k - \overline{c_k} z^{-k}\right)\right),$$

where c_n's are defined in (8). Hence, if v is a Beurling weight, then

$$\log w \in \mathcal{A}_v \quad \Leftrightarrow \quad \log \mathcal{S} \in \mathcal{A}_v \quad \Leftrightarrow \quad \text{both } \mathcal{S} \text{ and } \mathcal{S}^{-1} \in \mathcal{A}_v. \tag{17}$$

In particular, if v is a *strong* Beurling weight, then from Baxter's Theorem it follows that

$$\log w \in \mathcal{A}_v \quad \Leftrightarrow \quad w \in \mathcal{A}_v \quad \Leftrightarrow \quad \log \mathcal{S} \in \mathcal{A}_v \quad \Leftrightarrow \quad \{\alpha\}_0^\infty \in l_v.$$

Regarding the case of a Beurling weight with $R > 1$, the situation is different. It was observed by Simon [14] that the following three statements are equivalent:

(i) the Fourier coefficients of $\log w$ exhibit an exponential decay;
(ii) the Fourier coefficients of w exhibit an exponential decay;
(iii) α_n exhibit an exponential decay.

As (17) shows, we may replace w by $\mathcal{S}$ in (i) and (ii).

However, the decay rates above are not necessary the same; typically the decay in (ii) and (iii) is faster than in (i). As examples in [14, Chapter 7] show, the decay rate in (ii) can be greater or less than in (iii).

Our goal is to investigate these rates more precisely, extending Baxter's theorem to the case of $R > 1$. Our main results are gathered in Theorems 2, 3 and 4.

Let us start with the following technical result:

Lemma 2. *Let $h \in \mathcal{A}_1$, and let v be a Beurling weight which increases on $\mathbb{Z}_+$. Assume that $\mathcal{P}_-(h) \in \mathcal{A}_v^-$. Then for a function g*

$$g \in \mathcal{A}_1^+ \text{ or } g \in \mathcal{A}_v^- \quad \Rightarrow \quad \mathcal{P}_-(gh) \in \mathcal{A}_v^- .$$

Proof. We begin by proving the first assertion. Set $h = \sum_{k\in\mathbb{Z}} h_k z^k$ and $g = \sum_{n\geqslant 0} g_n z^n$. Then

$$\mathcal{P}_-(gh) = \sum_{i\geqslant 0} d_i z^{-i} \quad \text{with } d_i = \sum_{n\geqslant 0} g_n h_{-n-i} .$$

Hence,

$$v(i)|d_i| \leqslant v(i) \sum_{n\geqslant 0} |g_n|\,|h_{-n-i}| \leqslant \sum_{n\geqslant 0} v(n+i)|g_n|\,|h_{-n-i}| .$$

Therefore, using Fubini's theorem we get that

$$\|\mathcal{P}_-(gh)\|_v \leqslant \|\mathcal{P}_-(h)\|_v \; \|g\|_1 .$$

For the second assertion, write $h_- = \mathcal{P}_-(h)$ and $h_+ = h - h_- \in \mathcal{A}_1^+$. Then $\mathcal{P}_-(gh_-) \in \mathcal{A}_v^-$ by assumption, and $\mathcal{P}_-(gh_+) \in \mathcal{A}_v^-$ by the previous argument. Since $\mathcal{P}_-(gh) = \mathcal{P}_-(gh_+) + \mathcal{P}_-(gh_-)$, the statement follows. $\square$

Corollary 1. *Let $g(z) = \sum_{k\in\mathbb{Z}} d_k z^k \in \mathcal{A}_1$, and let v be a Beurling weight which increases on $\mathbb{Z}_+$. For a rational function r denote $r(z)g(z) = \sum_{k\in\mathbb{Z}} \hat{d}_k z^k$. If r has no poles in $1/R \leqslant |z| \leqslant 1$ then*

$$\sum_{k<0} d_k z^k \in \mathcal{A}_v^- \;\Rightarrow\; \sum_{k<0} \hat{d}_k z^k \in \mathcal{A}_v^- .$$

Let us recall some facts that we will need further (see [8] for details). If

$$\rho^{-1} \stackrel{\text{def}}{=} \overline{\lim_{n\to\infty}} |\alpha_n|^{1/n} < 1,$$

then (see [9]) also

$$\rho^{-1} = \inf\{0 < r < 1 : \; f_-(z) \text{ is holomorphic in } |z| > r\}.$$

For the scattering function $\mathcal{S}$ defined in (16), let

$$\mathcal{S}(z) = \sum_{k \in \mathbb{Z}} d_k z^k$$

be its Laurent expansion. An important fact, established independently in [2,13,8] (cf. Corollary 2 therein) is that for $n \in \mathbb{N}$,

$$\overline{\alpha_n} = -d_{-n-1} + e_n \quad \text{with } |e_n| \leqslant C\, r^{3n}, \tag{18}$$

for an arbitrary r such that $1/\rho < r \leqslant 1$, and where $C > 0$ does not depend on r or n.

Remark 2. Since $|\mathcal{S}(z)| = 1$ for $|z| = 1$, we may state this result equivalently in terms of the coefficients of $\mathcal{P}_+(1/\mathcal{S})$, see [7].

The above formulas indicate the close connection between the rate of decay of α_n's and the decay of the negative Fourier coefficients of the scattering function $\mathcal{S}$, which is made explicit in:

Theorem 2. *Let $d\mu = w(\theta)\, d\theta$ with $w \in \mathcal{A}_1$ and v be a Beurling weight. Then*

$$\alpha \in \ell_v \quad \Leftrightarrow \quad \mathcal{P}_-(\mathcal{S}) \in \mathcal{A}_v^-.$$

Proof. Since for any Beurling weight, $\mathcal{A}_v \subset \mathcal{A}_1$, Baxter's theorem (Theorem 1) shows that $f_\pm$ exist, $f_\pm \in \mathcal{A}_1^\pm$, and therefore $\mathcal{S} \in \mathcal{A}_1$. Furthermore, for any strong Beurling weight the results follow from Theorem 1 and Lemma 2, so we consider only the case when $R > 1$.

Assume that $\alpha \in \ell_v$, then $1 < R \leqslant \rho$. For an sufficiently small $\varepsilon > 0$ let us take $r = (1+\varepsilon)/R \in (1/\rho, 1)$. Since we may take r arbitrary close to $1/R$, assumption $\alpha \in \ell_v$ and (18) imply that $\{d_k\}_{k<0} \in \ell_v$, so that $\mathcal{P}_-(\mathcal{S}) \in \mathcal{A}_v^-$.

For converse, if $\mathcal{P}_-(\mathcal{S}) \in \mathcal{A}_v^-$, then Lemma 2 implies that $f_- \in \mathcal{A}_v^-$, showing that again $\rho > 1$. Now an application of formula (18) gives the assertion. $\square$

Remark 3. Given a space X of functions on $\mathbb{T}$, the *recovery problem* studies under which natural assumptions on $\mathcal{S}$ the implication $\mathcal{P}_-(\mathcal{S}) \in X \Rightarrow \mathcal{S} \in X$ holds. Many results for a wide class of spaces X, including algebras of functions, have been found in [12, Chapter 3] (see also [11, Chapters 7 and 13]).

A consequence of the previous theorem is the following result which shows that the product of two measures whose coefficients are in $\mathcal{A}_v$ is a new measure whose coefficients are also in $\mathcal{A}_v$.

Theorem 3. *For $d\mu_i = w_i(\theta)\, d\theta$, $i = 1, 2, 3$, with $w_i \in \mathcal{A}_1$ denote by $\alpha^{(i)}$ the corresponding sequences of the recurrence coefficients. Assume that $w_3 = w_1 w_2$, and let v be a Beurling weight. Then*

$$\alpha^{(1)}, \alpha^{(2)} \in \ell_v \quad \Rightarrow \quad \alpha^{(3)} \in \ell_v.$$

Proof. Let us denote by $\mathcal{S}_i$ the scattering function corresponding to μ_i. By Baxter's theorem, $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{A}_1$, and by Theorem 2, for $i = 1, 2$,

$$\alpha^{(i)} \in \ell_v \quad \Rightarrow \quad \mathcal{S}_i^- \stackrel{\text{def}}{=} \mathcal{P}_-(\mathcal{S}_i) \in \mathcal{A}_v^-, \quad \mathcal{S}_i^+ \stackrel{\text{def}}{=} \mathcal{S}_i - \mathcal{S}_i^- \in \mathcal{A}_1^+.$$

Hence,

$$\mathcal{P}_-(\mathcal{S}_3) = \mathcal{P}_-(\mathcal{S}_1\mathcal{S}_2) = \mathcal{P}_-(\mathcal{S}_1^-\mathcal{S}_2^- + \mathcal{S}_1^+\mathcal{S}_2^- + \mathcal{S}_1^-\mathcal{S}_2^+) \in \mathcal{A}_v^-,$$

where we have used Lemma 2. The result now follows from Theorem 2. □

Now we are in the position of discussing several corollaries of the previous results.

Corollary 2. *Let v be a Beurling weight, and for the decomposition* (7), $d\mu_s = 0$ *and* $w \in \mathcal{A}_v$. *If* $w(z) \neq 0$ *in* $R^{-1} \leqslant |z| \leqslant R$, *where R is defined in* (6), *then* $\{\alpha\}_0^\infty \in \ell_v$.

This result for complex weights with zero winding numbers has been proved in [2].

Proof. According to (17) and Theorem 2, under assumptions of the corollary,

$$\log w \in \mathcal{A}_v \quad\Rightarrow\quad \log \mathcal{S} \in \mathcal{A}_v \quad\Rightarrow\quad \mathcal{S} \in \mathcal{A}_v \quad\Rightarrow\quad \mathcal{P}_-(\mathcal{S}) \in \mathcal{A}_v \quad\Rightarrow\quad \alpha \in \ell_v. \qquad \square$$

Another straightforward consequence of Theorem 3 is that the multiplication of the original weight by a Bernstein–Szegő weight does not affect the rate of decay of the recurrence coefficients.

Corollary 3. *Let* $d\mu = w(\theta)\, d\theta$, *with* $w \in \mathcal{A}_1$, *and let p be a polynomial with no zeros on* $\mathbb{T}$. *Denote by* α *and* $\hat{\alpha}$ *the sequences of the recurrence coefficients corresponding to measures* $d\mu = w(\theta)\, d\theta$ *and* $d\hat{\mu} = |p|^{-2} w(\theta)\, d\theta$, *respectively. Then for any Beurling weight v*,

$$\alpha \in \ell_v \quad\Rightarrow\quad \hat{\alpha} \in \ell_v.$$

The proof is an immediate consequence of Theorem 3, by observing that for the weight $|p|^{-2}$ the corresponding recurrence coefficients vanish for a sufficiently large n, and hence belong to ℓ_v for any Beurling weight v.

Finally we formulate an extension of Baxter's theorem (Theorem 1) for the case $R > 1$.

Theorem 4. *Suppose that v is a Beurling weight with* $R > 1$.

If there exists a polynomial $p(z) \neq 0$ *for* $|z| = 1$, *such that for*

$$\hat{w}(z) \stackrel{def}{=} |p(z)|^2 w(z), \quad |z| = 1, \tag{19}$$

we have $\log(\hat{w}) \in \mathcal{A}_v$, *then* $\{\alpha\}_0^\infty \in \ell_v$.

Conversely, if $\{\alpha\}_0^\infty \in \ell_v$ *and* $1/w \neq 0$ *for* $|z| = R$ *then there exists a polynomial p such that for* $\hat{w}(z)$ *defined by (19) we have* $\log(\hat{w}) \in \mathcal{A}_v$.

Proof. Let us prove the first implication. Assume that $\log(\hat{w}) \in \mathcal{A}_v$. By Corollary 2, if $\hat{\alpha}$ are the recurrence coefficients corresponding to $\hat{w}$, then $\{\hat{\alpha}\}_0^\infty \in \ell_v$. But $w = |p|^{-2}\hat{w}$, and it remains to apply Corollary 3.

Conversely, let $\{\alpha\}_0^\infty \in \ell_v$. Since by Lemma 1, $1/w = f_+ f_- \in \mathcal{A}_v$, and $1/w(z) \neq 0$ for $|z| = R$, then f_+ has a finite number of zeros, $\zeta_1, \ldots, \zeta_m$, in $1 < |z| < R$ (denoted with account of their multiplicity). Let

$$p(z) = \prod_{k=1}^{m} (z - \zeta_k).$$

Then $\hat{w}$ satisfies the conditions of Corollary 2, which concludes the proof. □

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